November, 2004

INTEGRABLE SYSTEMS AND DISCRETE GEOMETRY

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1 Introduction

Although the main subject of this article is the connection between integrable discrete systems and geometry, we feel obliged to begin with the differential part of the relation.

1.1 Classical differential geometry and integrable systems

The oldest (1840) integrable nonlinear partial differential equation recorded in literature is the Lamé system

$$\frac{\partial^2 H_i}{\partial u_j \partial u_k} - \frac{1}{H_j}\frac{\partial H_j}{\partial u_k}\frac{\partial H_i}{\partial u_j} - \frac{1}{H_k}\frac{\partial H_k}{\partial u_j}\frac{\partial H_i}{\partial u_k} = 0, \qquad i,j,k \quad \text{distinct} \tag{1.1}$$

$$\frac{\partial}{\partial u_k}\left(\frac{1}{H_k}\frac{\partial H_j}{\partial u_k}\right) + \frac{\partial}{\partial u_j}\left(\frac{1}{H_j}\frac{\partial H_k}{\partial u_j}\right) + \frac{1}{H_i^2}\frac{\partial H_j}{\partial u_i}\frac{\partial H_k}{\partial u_i} = 0, \tag{1.2}$$

describing orthogonal coordinates in the three dimensional Euclidean space $\mathbb{E}^3$ (indices i,j,k range from 1 to 3). Already in 1869 it was found by Ribaucour that the *nonlinear* Lamé system possesses a discrete symmetry enabling to construct, in a *linear* way, new solutions of the system from the old ones. He gave also a geometric interpretation of this symmetry in terms of certain spheres tangent to the coordinate surfaces of the triply orthogonal system. In 1918 Bianchi showed that the result of superposition of the Ribaucour transformations is, in a certain sense, independ of the order of their composition.

Such properties of a nonlinear equation are hallmarks of its integrability, and indeed, the Lamé system was solved using soliton techniques in 1997-98. The above example illustrates the close connection between the modern theory of integrable partial differential equations and the differential geometry of the turn of the XIXth and XXth centuries. A remarkable property of certain parametrized submanifolds (and then of the corresponding equations) studied that time is that they allow for transformations which exhibit the so called *Bianchi permutability property*. Such transformations called, depending on the context, the Darboux, Calapso, Christoffel, Bianchi, Bäcklund, Laplace, Koenigs, Moutard, Combescure, Lévy, Goursat, Ribaucour or the fundamental transformation of Jonas, can be geometrically described in terms of certain families of lines called line congruences.

In the connection between integrable systems and differential geometry, a distinguished role is played by the multidimensional conjugate nets, described by the Darboux system, which is just the first part (1.1) of the Lamé system with indices ranging form 1 to $N \geq 3$. On the level of integrable systems, this dominant role has the following explanation: the Darboux system, together with equations describing iso-conjugate deformations of the net, forms the multicomponent Kadomtsev–Petviashvilii (KP) hierarchy, which is viewed as a master system of equations in soli-

ton theory. In fact, in appropriate variables, the whole multicomponent KP hierarchy can be rewritten as an infinite system of the Darboux equations.

1.2 Transition to the discrete domain

The recent progress in studying discrete integrable systems showed that, in many respects, they should be considered as more fundamental than their differential counterparts. Consequently, the natural problem of extending the geometric interpretation of integrable partial differential equations to the discrete domain arose, leading not only to the transition to the discrete domain of many results on the connection between the differential geometry and integrable systems, but also – and this seems to be even more important – to the description of integrability in a very elementary and purely geometric way.

On the level of integrable equations, the transition 'from differential to discrete' often makes formulas more complicated and longer. On the contrary, on the geometric level, in such a transition the properties of discrete submanifolds, relevant to their integrability, become simpler and more transparent. Indeed, the mathematics necessary to understand the basic ideas of the integrable discrete geometry does not exceed the 'ruler and compass constructions', and many proofs can be performed using elementary incidence geometry.

We will concentrate our attention on the multidimensional lattice made from planar quadrilaterals, which is the discrete analogue of a conjugate net. Together with the discussion of its properties, which are the core of the geometric integrability, we briefly present the analytic methods of construction of these lattices and we also describe some basic multidimensional integrable reductions of them. Then we discuss integrable discrete surfaces; some of them have been found in the early period of the 'case by case' studies. We shall however try to present them, from a unifying perspective, as reductions of the quadrilateral lattice.

2 Multidimensional integrable lattices

2.1 The quadrilateral lattice

An N dimensional lattice $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^M$ is a lattice made from planar quadrilaterals, or a quadrilateral lattice (QL) in short, if its elementary quadrilaterals $\{\boldsymbol{x}, T_i\boldsymbol{x}, T_j\boldsymbol{x}, T_iT_j\boldsymbol{x}\}$ are planar; i.e., iff the following system of discrete Laplace equations is satisfied

$$\Delta_i\Delta_j\boldsymbol{x} = (T_iA_{ij})\Delta_i\boldsymbol{x} + (T_jA_{ji})\Delta_j\boldsymbol{x}, \quad i \neq j, \quad i,j = 1,\dots,N, \tag{2.1}$$

where $A_{ij} : \mathbb{Z}^N \to \mathbb{R}$ are functions of the discrete variable; here T_i is the translation operator in the ith direction and $\Delta_i = T_i - 1$ is the corresponding difference opera-

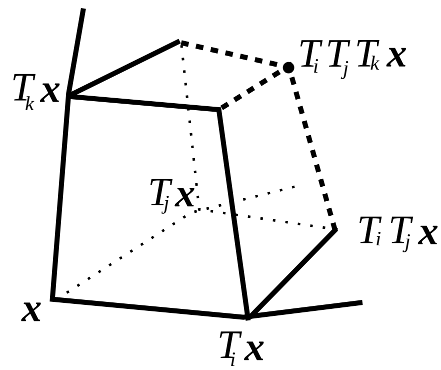

Figure 1: The geometric integrability scheme

tor. For simplicity we work here in the affine setting neglecting projective geometric aspects of the theory.

2.1.1 The geometric integrability scheme

In the case $N = 2$ the definition (2.1) allows one to uniquely construct, given two discrete curves intersecting in a common vertex and two functions $A_{12}, A_{21} : \mathbb{Z}^2 \to \mathbb{R}$, a quarilateral surface. For $N > 2$ the planarity constraints (2.1) are instead compatible if and only if the geometric data A_{ij} satisfy the nonlinear system

$$\Delta_k A_{ij} + (T_k A_{ij})A_{ik} = (T_j A_{jk})A_{ij} + (T_k A_{kj})A_{ik}, \qquad i, j, k \quad \text{distinct.} \tag{2.2}$$

This constraint has very simple interpretation: in building the elementary cube (see Figure 1), the seven points $\boldsymbol{x}$, $T_i\boldsymbol{x}$, $T_j\boldsymbol{x}$, $T_k\boldsymbol{x}$, $T_iT_j\boldsymbol{x}$, $T_iT_k\boldsymbol{x}$ and $T_jT_k\boldsymbol{x}$ (i, j, k are distinct) determine the eighth point $T_iT_jT_k\boldsymbol{x}$ as the unique intersection of three planes in the three dimensional space.

The connection of this elementary geometric point of view with the classical theory of integrable systems is transparent: the planarity constraint corresponds to the set of *linear* spectral problems (2.1) and the resulting QL is characterized by the nonlinear equations (2.2), arising as the compatibility conditions for such spectral problems. Since the QL equations (2.2) are a master system in the theory of integrable equations, *planarity* can be viewed as the *elementary geometric root of integrability*. The idea that integrability be associated with the consistency of a geometric (and/or algebraic) property when increasing the dimensionality of the system is recurrent in the theory of integrable systems.

2.1.2 Other forms of the Darboux system

The $i \leftrightarrow j$ symmetry of the RHS of equations (2.2) implies the existence of the potentials $H_i : \mathbb{Z}^N \to \mathbb{R}$ (the Lamé coefficients) such that

$$A_{ij} = \frac{\Delta_j H_i}{H_i}, \qquad i \neq j, \tag{2.3}$$

and then equations (2.2) take the form

$$\Delta_k \Delta_j H_i - \left(T_j \frac{\Delta_k H_j}{H_j} \right) \Delta_j H_i - \left(T_k \frac{\Delta_j H_k}{H_k} \right) \Delta_k H_i = 0, \qquad i, j, k \quad \text{distinct}, \tag{2.4}$$

being the discrete version of the first part (1.1) of the Lamé system.

The Lamé coefficients allow to define the suitably normalized tangent vectors $\boldsymbol{X}_i : \mathbb{Z}^N \to \mathbb{R}^M$ by equations

$$\Delta_i \boldsymbol{x} = (T_i H_i) \boldsymbol{X}_i, \tag{2.5}$$

and the functions $Q_{ij} : \mathbb{Z}^N \to \mathbb{R}$, $i \neq j$, (the rotation coefficients) by equations

$$\Delta_i H_j = (T_i H_i) Q_{ij}, \qquad i \neq j. \tag{2.6}$$

Then equations (2.1) and (2.4) can be rewritten in the first order form

$$\Delta_j \boldsymbol{X}_i = (T_j Q_{ij}) \boldsymbol{X}_j, \qquad i \neq j, \tag{2.7}$$

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj}, \qquad i, j, k \quad \text{distinct}. \tag{2.8}$$

The discrete Darboux system (2.8) implies the existence of other potentials ρ_i defined by the compatible equations

$$\frac{T_j \rho_i}{\rho_i} = 1 - (T_i Q_{ji})(T_j Q_{ij}), \qquad i \neq j. \tag{2.9}$$

The $i \leftrightarrow j$ symmetry of the RHS of equations (2.9) implies existence of yet another potential $\tau : \mathbb{Z}^N \to \mathbb{R}$,

$$\rho_i = \frac{T_i \tau}{\tau}, \tag{2.10}$$

which is called the τ-function of the quadrilateral lattice. In terms of the τ function, and of the functions

$$\tau_{ij} = \tau Q_{ij}, \qquad i \neq j, \tag{2.11}$$

whose geometric interpretation will be given in Section 2.3.3, the discrete Darboux equations take the following Hirota-type form

$$(T_i T_j \tau)\tau = (T_i \tau) T_j \tau - (T_i \tau_{ji}) T_j \tau_{ij}, \qquad i \neq j, \tag{2.12}$$

$$(T_k \tau_{ij})\tau = (T_k \tau)\tau_{ij} + (T_k \tau_{ik})\tau_{kj}, \qquad i, j, k \quad \text{distinct}. \tag{2.13}$$

2.2 Analytic methods

We will show how one can construct large classes of solutions of the discrete Darboux equations and the corresponding quadrilateral lattices using two basic analytical methods of the soliton theory: the $\bar{\partial}$ dressing method and the algebro-geometric techniques.

2.2.1 The $\bar{\partial}$ dressing method

Consider the non-local $\bar{\partial}$-problem

$$\bar{\partial}\chi(z) + (\hat{R}\chi)(z) = \bar{\partial}\nu(z), \qquad \lim_{|z|\to\infty}(\chi(z) - \nu(z)) = 0, \tag{2.14}$$

where $\bar{\partial} = \partial/\partial\bar{z}$, $\hat{R}$ is the integral operator

$$(\hat{R}\chi)(z) = \int_{\mathbb{C}} R(z,z')\chi(z')\, dz' \wedge d\bar{z}',$$

and $\nu(z)$ is a given rational function of z.

Let $Q_i^{\pm} \in \mathbb{C}$, $i = 1, \ldots, N$ be pairs of distinct points of the complex plane, which define the dependence of the kernel R on the discrete variable $n \in \mathbb{Z}^N$

$$R(z,z';n) = \prod_{i=1}^{N}\left(\frac{z - Q_i^+}{z - Q_i^-}\right)^{n_i} R_0(z,z') \prod_{i=1}^{N}\left(\frac{z' - Q_i^-}{z' - Q_i^+}\right)^{n_i}.$$

We consider only kernels $R_0(z,z')$ such that the non-local $\bar{\partial}$-problem is uniquely solvable. If $\chi(z;n)$ is the unique solution with the canonical normalization $\nu = 1$, then the function

$$\psi(z;n) = \chi(z;n)\prod_{i=1}^{N}\left(\frac{z - Q_i^-}{z - Q_i^+}\right)^{n_i}$$

satisfies the system of the Laplace equations (2.1) with the Lamé coefficients given by

$$H_i(n) = \lim_{z\to Q_i^+}\left(\left(\frac{z - Q_i^+}{z - Q_i^-}\right)^{n_i}\psi(z;n)\right).$$

By construction, the system of such Laplace equations is compatible, therefore the Lamé coefficients satisfy equations (2.4). To various n-independent measures $d\mu_a$ on $\mathbb{C}$ there correspond coordinates

$$x^a(n) = \int_{\mathbb{C}} \psi(z;n) d\mu_a(z),$$

of a quadrilateral lattice $\boldsymbol{x}$, having $H_i(n)$ as the Lamé coefficients. To have real lattices, the kernel R_0, the points $Q_i^{\pm}$ and the measures $d\mu_a$ should satisfy certain additional conditions.

One can find a similar interpretation of the normalized tangent vectors $\boldsymbol{X}_i$ and of the rotation coefficients Q_{ij}. If $\chi_i(z;n)$ are the unique solutions of the non-local $\bar{\partial}$-problem (2.14) with the normalizations

$$\nu_i(z;n) = \left(\frac{Q_i^+ - Q_i^-}{z - Q_i^+}\right) \prod_{k=1, k\neq i}^{N} \left(\frac{Q_i^+ - Q_k^+}{Q_i^+ - Q_k^-}\right)^{n_k},$$

then the functions $\psi_i(z;n)$, defined by

$$\psi_i(z;n) = \prod_{k=1}^{N} \left(\frac{z - Q_k^-}{z - Q_k^+}\right)^{n_k} \chi_i(z;n),$$

satisfy the direct analogue of the linear problem (2.7)

$$\Delta_j \psi_i(z;n) = (T_j Q_{ij}(n)) \psi_j(z;n), \qquad i \neq j, \tag{2.15}$$

where

$$Q_{ij}(n) = \lim_{z \to Q_j^+} \left(\left(\frac{z - Q_j^+}{z - Q_j^-}\right)^{n_j} \psi_i(z;n) \right).$$

Again, by construction, equations (2.15) are compatible and the functions $Q_{ij}(n)$ satisfy the discrete Darboux equations (2.8). The functions

$$X_i^a(n) = \int_{\mathbb{C}} \psi_i(z;n) d\mu_a(z),$$

are coordinates of the normalized tangent vectors $\boldsymbol{X}_i$ of the quadrilateral lattice $\boldsymbol{x}$ constructed above.

2.2.2 The algebro-geometric techniques

Given a compact Riemann surface $\mathcal{R}$ of genus g, consider a non-special divisor $D = \sum_{\alpha=1}^{g} P_\alpha$. Choose N pairs of points $Q_i^{\pm} \in \mathcal{R}$ and the normalization point Q_∞. Given $n \in \mathbb{Z}^N$, there exists a unique Baker–Akhiezer function $\psi(n)$, defined as a meromorphic function on $\mathcal{R}$, with the following analytical properties: (i) as a function of $P \in \mathcal{R} \setminus \cup_{i=1}^{N} Q_i^{\pm}$, $\psi(n)$ may have as singularities only simple poles in the points of the divisor D; (ii) in the points $Q_i^{\pm}$ function $\psi(n)$ has poles of the order $\pm n_i$; (iii) in the point Q_∞ function $\psi(n)$ is normalized to 1.

When $z_i^\pm(P)$ is a local coordinate on $\mathcal{R}$ centered at $Q_i^\pm$, then condition (ii) implies that the function $\psi(n)$ in a neigbourhood of the point $Q_i^\pm$ is of the form

$$\psi(P;n) = \left(z_i^\pm(P)\right)^{\mp n_i} \left(\sum_{s=0}^{\infty} \xi^i_{s,\pm}(n) \left(z_i^\pm(P)\right)^s \right). \tag{2.16}$$

The Baker–Akhiezer function, as a function of the discrete variable $n \in \mathbb{Z}^N$, satisfies the system of Laplace equations (2.1) with the Lamé coefficients $H_i(n) = \xi^i_{0,+}(n)$.

Again, by construction, the Lamé coefficients satisfy equations (2.4). To various n-independent measures $d\mu_a$ on $\mathcal{R}$ there correspond coordinates

$$x^a(n) = \int_{\mathcal{R}} \psi(P;n) d\mu_a(P),$$

of a quadrilateral lattice $\boldsymbol{x}$.

We present the expression of the Baker–Akhiezer function and of the τ-function of the quadrilateral lattice in terms of the Riemann theta functions. Let us choose on $\mathcal{R}$ the canonical basis of cycles $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ and the dual basis $\{\omega_1, \ldots, \omega_g\}$ of holomorphic differentials on $\mathcal{R}$, i.e., $\oint_{a_j} \omega_k = \delta_{jk}$.Then the matrix B of b-periods defined as $B_{jk} = \oint_{b_j} \omega_k$,is symmetric and has positively defined imaginary part. Denote by ω_{PQ} the unique differential holomorphic in $\mathcal{R} \setminus \{P, Q\}$ with poles of the first order in P, Q and residues, correspondingly, 1 and -1, which is normalized by conditions $\oint_{a_j} \omega_{PQ} = 0$.The Riemann function $\theta(z; B)$, $z \in \mathbb{C}^g$, is defined by its Fourier expansion

$$\theta(z;B) = \sum_{m \in \mathbb{Z}^g} \exp\left\{ \pi i \langle m, Bm \rangle + 2\pi i \langle m, z \rangle \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard bilinear form in $\mathbb{C}^g$. Finally, the Abel map A is given by $A(P) = \left(\int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g \right)$, where $P_0 \in \mathcal{R}$, and the Riemann constants vector K is given by

$$K_j = \frac{1 + B_{jj}}{2} - \sum_{k \neq j} \left(\oint_{a_k} \omega_k(P) A_j(P) \omega_j \right), \qquad j = 1, \ldots, g.$$

The explicit form of the vacuum Baker–Akhiezer function ψ can be written down with the help of the theta functions as follows

$$\psi(n,P) = \frac{\theta\left(A(P) + \sum_{k=1}^N n_k \left(A(Q_k^-) - A(Q_k^+)\right) + Z\right)}{\theta\left(A(Q_\infty) + \sum_{k=1}^N n_k \left(A(Q_k^-) - A(Q_k^+)\right) + Z\right)} \times \\ \times \frac{\theta\left(A(Q_\infty) + Z\right)}{\theta\left(A(P) + Z\right)} \exp\left(\sum_{k=1}^N n_k \int_{Q_\infty}^P \omega_{Q_k^- Q_k^+} \right),$$

where $Z = -\sum_{j=1}^{g} A(P_j) - K$.

Denote by $r^{\pm}_{kj}$ and $s^{\pm}_{kj}$ the constants in the decomposition of the Abelian integrals near the point $Q^{\pm}_j$

$$\int_{P_0}^{P} \omega_{Q_k^- Q_k^+} \stackrel{P\to Q_j^{\pm}}{=} \mp\delta_{kj} \log z_j^{\pm}(P) + r^{\pm}_{kj} + O(z_j^{\pm}(P)),$$
$$\int_{P_0}^{P} \omega_{Q_\infty Q_k^+} \stackrel{P\to Q_j^{\pm}}{=} -\delta_{kj}\delta_{+\pm} \log z_j^{\pm}(P) + s^{\pm}_{kj} + O(z_j^{\pm}(P)).$$

Then the expression of the τ–function of the quadrilateral lattice within the subclass of algebro-geometric solutions reads

$$\tau(n) = \theta\left(\sum_{k=1}^{N} n_k \left(A(Q_k^-) - A(Q_k^+) \right) + A(Q_\infty) + Z \right) \prod_{k,j=1}^{N} \lambda_{kj}^{n_k n_j} \prod_{k=1}^{N} \mu_k^{n_k},$$

where

$$\lambda_{kj} = \exp\left(\frac{r^-_{kj} - r^+_{kj}}{2} \right) = \lambda_{jk}, \qquad \mu_k = \frac{1}{\lambda_{kk}} \frac{\theta\left(A(Q_k^+) + Z\right)}{\theta\left(A(Q_k^-) + Z\right)} \exp\left(s^-_{kk} - s^+_{kk} \right).$$

Finally, we remark that the geometric integrability scheme and the algebro-geometric methods work also in the finite fields setting, giving solutions of the corresponding integrable cellular automata.

2.3 The Darboux-type transformations

We present the basic ideas and results of the theory of the Darboux type transformations of the multidimensional quadrilateral lattice.

2.3.1 Line congruences and the fundamental transformation

To define the transformations we need to define first N-dimensional *line congruences* (or, simply, congruences), which are familes of lines in $\mathbb{R}^M$ labelled by points of $\mathbb{Z}^N$ with the property that any two neighbouring lines $\mathfrak{l}$ and $T_i\mathfrak{l}$, $i = 1, ..., N$, are coplanar and therefore (eventually in the projective extension $\mathbb{P}^M$ of $\mathbb{R}^M$) intersect.

The quadrilateral lattice $\mathcal{F}(\boldsymbol{x})$ is a *fundamental transform* of the quadrilateral lattice $\boldsymbol{x}$ if the lines connecting the corresponding points of the lattices form a congruence. The superposition of a number of fundamental transformations can be compactly formulated in the vectorial fundamental transformation. The data of the vectorial fundamental transformation are: (i) the solution $\boldsymbol{Y}_i : \mathbb{Z}^N \to \mathbb{V}$, $\mathbb{V}$ being a linear space, of the linear system (2.7); (ii) the solution $\boldsymbol{Y}^*_i : \mathbb{Z}^N \to \mathbb{V}^*$, $\mathbb{V}^*$ being the dual of $\mathbb{V}$, of the linear system (2.6). These allow to construct the linear operator valued

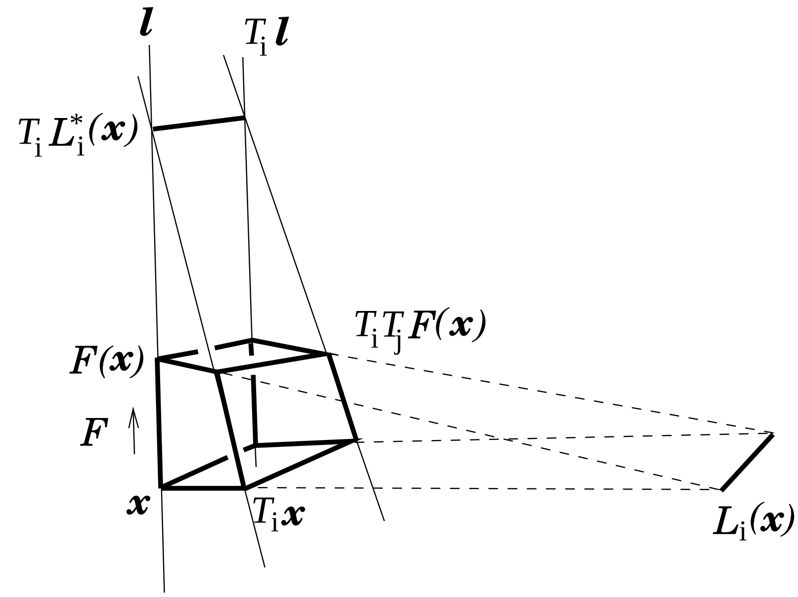

Figure 2: The fundamental transformation as the binary transformation

potential $\boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*) : \mathbb{Z}^N \to L(\mathbb{V})$, defined by the following analogue of equation (2.5)

$$\Delta_i \boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*) = \boldsymbol{Y}_i \otimes (T_i \boldsymbol{Y}^*_i), \qquad i = 1, \dots, N; \tag{2.17}$$

similarly, one defines $\boldsymbol{\Omega}(\boldsymbol{X},\boldsymbol{Y}^*) : \mathbb{Z}^N \to L(\mathbb{V},\mathbb{R}^M)$ and $\boldsymbol{\Omega}(\boldsymbol{Y},H) : \mathbb{Z}^N \to \mathbb{V}$. The transforms of the lattice $\boldsymbol{x}$ and other related functions are given by

$$\begin{aligned}
\boldsymbol{\mathcal{F}}(\boldsymbol{x}) &= \boldsymbol{x} - \boldsymbol{\Omega}(\boldsymbol{X},\boldsymbol{Y}^*)\boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*)^{-1}\boldsymbol{\Omega}(\boldsymbol{Y},H),\\
\boldsymbol{\mathcal{F}}(H_i) &= H_i - \boldsymbol{Y}^*_i\boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*)^{-1}\boldsymbol{\Omega}(\boldsymbol{Y},H), \qquad i = 1,\dots,N,\\
\boldsymbol{\mathcal{F}}(\boldsymbol{X}_i) &= \boldsymbol{X}_i - \boldsymbol{\Omega}(\boldsymbol{X},\boldsymbol{Y}^*)\boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*)^{-1}\boldsymbol{Y}_i, \qquad i = 1,\dots,N,\\
\boldsymbol{\mathcal{F}}(Q_{ij}) &= Q_{ij} - \boldsymbol{Y}^*_j\boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*)^{-1}\boldsymbol{Y}_i, \qquad i,j = 1,\dots,N, \quad i \neq j,\\
\boldsymbol{\mathcal{F}}(\rho_i) &= \rho_i \left(1 + (T_i\boldsymbol{Y}^*_i)\boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*)\boldsymbol{Y}_i\right), \qquad i = 1,\dots,N,\\
\boldsymbol{\mathcal{F}}(\tau) &= \tau \det \boldsymbol{\Omega}(\boldsymbol{Y},\boldsymbol{Y}^*).
\end{aligned}$$

Notice that, by the coplanarity of any two neighbouring lines of the congruence, also the quadrilaterals $\{\boldsymbol{x}, T_i\boldsymbol{x}, \mathcal{F}(\boldsymbol{x}), \mathcal{F}(T_i\boldsymbol{x})\}$ are planar (see Figure 2). Then the construction of the transformed lattice mimics the geometric integrability scheme. In consequence, any quadrilateral $\{\boldsymbol{x}, \mathcal{F}_1(\boldsymbol{x}), \mathcal{F}_2(\boldsymbol{x}), \mathcal{F}_1(\mathcal{F}_2(\boldsymbol{x})) = \mathcal{F}_2(\mathcal{F}_1(\boldsymbol{x}))\}$ is planar as well. Therefore, on the discrete level, there is no difference between the lattice coordinate directions and the fundamental transformation directions. The distinction becomes visible in the limit from the quadrilateral lattice to the conjugate net. Therefore the vectorial description of the superposition of the fundamental transformations

not only implies their permutability but also provides the explanation of the validity of the practical rule of "integrable discretization by Darboux transformations".

2.3.2 The Lévy and Combescure transformations

It is easy to see that the family $\mathfrak{t}_i$ of lines passing through the points $\boldsymbol{x}$ and $T_i\boldsymbol{x}$ of a quadrilateral lattice forms a congruence, called the i-th *tangent congruence* of the lattice. When the congruence of the transformation is the i-th tangent congruence of the lattice $\boldsymbol{x}$, then the corresponding reduction of the fundamental transformation is called the *Lévy transformation* $\mathcal{L}_i$.

It turns out that, for a generic congruence $\mathfrak{l}$, the lattice made from intersection points of the lines $\mathfrak{l}$ and $T_i^{-1}\mathfrak{l}$ is a quadrilateral lattice, called the i-th *focal latice* of the congruence. When the fundamental transform of the lattice $\boldsymbol{x}$ is the i-th focal lattice of the transformation congruence, then the corresponding reduction of the fundamental transformation is called the *adjoint Lévy transformation* $\mathcal{L}_i^*$.

Both Lévy transformations use only a half of the fundamental transformation data, and the corresponding reduction formulas (in the scalar case) for the lattice points read as follows

$$\mathcal{L}_i(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{X}_i\,(Y_i)^{-1}\boldsymbol{\Omega}(Y,H),$$
$$\mathcal{L}_i^*(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{\Omega}(\boldsymbol{X},Y^*)\,(Y_i^*)^{-1}H_i.$$

Notice that the composition of the Lévy and the adjoint Lévy transformations gives (see Figure 2) the fundamental transformation, also called, for this reason, the binary transformation.

Another reduction of the fundamental transformation, important from a technical point of view, is the *Combescure transformation*, in which the tangent lines of the transformed lattice $\mathcal{C}(\boldsymbol{x})$ are parallel to those of the lattice $\boldsymbol{x}$. The transformation formula reads

$$\mathcal{C}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{\Omega}(\boldsymbol{X},Y^*),$$

where only the solution Y^* of the adjoint linear system (2.6), necessary to build the transformation congruence, is needed.

2.3.3 The Laplace transformations and the geometric meaning of the Hirota equation

The Laplace transform $\mathcal{L}_{ij}(\boldsymbol{x})$, $i \neq j$, of the quadrilateral lattice $\boldsymbol{x}$ is the j-th focal lattice of its i-th tangent congruence (see Figure 3). It is uniquely determined once the lattice $\boldsymbol{x}$ is given. The transformation formulas of the lattice points and of the τ-function read as follows

$$\mathcal{L}_{ij}(\boldsymbol{x}) = \boldsymbol{x} - \frac{1}{A_{ji}}\Delta_i\boldsymbol{x}, \tag{2.18}$$

$$\mathcal{L}_{ij}(\tau) = \tau_{ij} = \tau Q_{ij}. \tag{2.19}$$

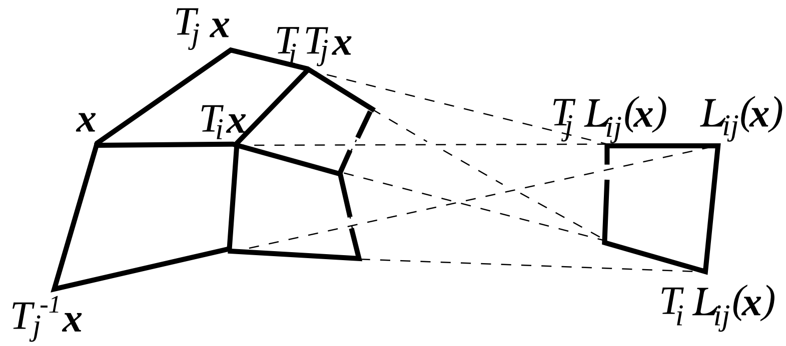

Figure 3: The Laplace transformation $\mathcal{L}_{ij}$

The superpositions of Laplace transformations satisfy the following identities

$$\mathcal{L}_{ij} \circ \mathcal{L}_{ji} = \mathrm{id}, \qquad \mathcal{L}_{jk} \circ \mathcal{L}_{ij} = \mathcal{L}_{ik}, \qquad \mathcal{L}_{ki} \circ \mathcal{L}_{ij} = \mathcal{L}_{kj},$$

which allow to identify them with the the Schlesinger transformations of the monodromy theory.

In the simplest case $N = 2$ one obtains the so called Laplace sequence of two dimensional quadrilateral lattices

$$\boldsymbol{x}_\ell = \mathcal{L}_{12}^\ell(\boldsymbol{x}), \qquad \tau_\ell = \mathcal{L}_{12}^\ell(\tau), \qquad \mathcal{L}_{12}^{-1} = \mathcal{L}_{21}, \quad \ell \in \mathbb{Z}.$$

Equations (2.12) and (2.19) imply that the τ-functions of the Laplace sequence satisfy the celebrated Hirota equation (the fully discrete Toda system)

$$\tau_\ell T_1 T_2 \tau_\ell = (T_1\tau_\ell)(T_2\tau_\ell) - (T_1\tau_{\ell-1})(T_2\tau_{\ell+1}).$$

2.4 Distinguished integrable reductions

We will present here basic reductions of the multidimensional quadrilateral lattice. The geometric criterion for their integrability is the compatibility with the geometric integrability scheme.

2.4.1 The circular lattices and the Ribaucour congruences

Quadrilateral lattices $\mathbb{Z}^N \to \mathbb{E}^M$ for which each quadrilateral is inscribed in a circle are called *circular* lattices. They are the integrable discrete analogues of submanifolds parametrized by curvature coordinates (for example, the orthogonal coordinate systems described by the Lamé equations (1.1)-(1.2)).

The integrability of circular lattices is the consequence of the fact that, if the three "initial" quadrilaterals $\{\boldsymbol{x}, T_i\boldsymbol{x}, T_j\boldsymbol{x}, T_iT_j\boldsymbol{x}\}$, $\{\boldsymbol{x}, T_i\boldsymbol{x}, T_k\boldsymbol{x}, T_iT_k\boldsymbol{x}\}$, $\{\boldsymbol{x}, T_j\boldsymbol{x}, T_k\boldsymbol{x}, T_jT_k\boldsymbol{x}\}$, are circular, then also the three new quadrilaterals constructed by adding the vertex

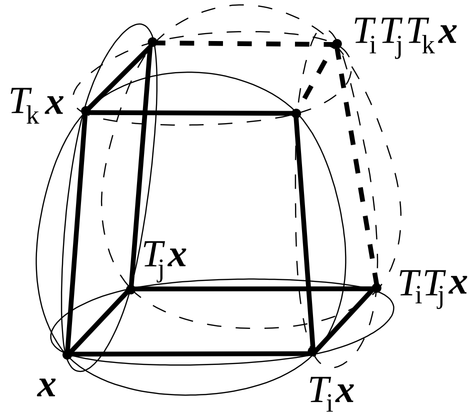

Figure 4: The geometric integrability of circular lattices

$T_iT_jT_k\boldsymbol{x}$, are circular as well (see Figure 4). In fact, all the eight vertices belong to a sphere, and, in consequence, all the vertices of any K dimensional, $K = 2, \ldots, N$, elementary cell belong to a $K-1$ dimensional sphere.

There are various equivalent algebraic descriptions of the circular lattices:
1. the normalized tangent vectors $\boldsymbol{X}_i$ satisfy the constraint

$$\boldsymbol{X}_i \cdot T_i\boldsymbol{X}_j + \boldsymbol{X}_j \cdot T_j\boldsymbol{X}_i = 0, \qquad i \neq j;$$

2. the scalar function $\boldsymbol{x}\cdot\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}$ satisfies the Laplace equations (2.1) of the lattice $\boldsymbol{x}$;
3. the functions $X_i^\circ = (\boldsymbol{x} + T_i\boldsymbol{x})\cdot\boldsymbol{X}_i : \mathbb{Z}^N \to \mathbb{R}$ satisfy the same linear system (2.7) as the normalized tangent vectors $\boldsymbol{X}_i$;
4. the functions $\boldsymbol{X}_i \cdot \boldsymbol{X}_i : \mathbb{Z}^N \to \mathbb{R}$ satisfy equations (2.9) and thus can serve as the potentials ρ_i.

The Ribaucour transformation $\mathcal{R}$ is the restriction of the fundamental transformation to the class of circular lattices such that also the "side" quadrilaterals $\{\boldsymbol{x}, T_i\boldsymbol{x}, \mathcal{R}(\boldsymbol{x}), \mathcal{R}(T_i\boldsymbol{x})\}$ are circular. Again there is no geometric difference between the lattice directions and the Ribaucour transformation direction. Moreover, the quadrilaterals $\{\boldsymbol{x}, \mathcal{R}_1(\boldsymbol{x}), \mathcal{R}_2(\boldsymbol{x}), \mathcal{R}_1(\mathcal{R}_2(\boldsymbol{x})) = \mathcal{R}_2(\mathcal{R}_1(\boldsymbol{x}))\}$ are circular as well. In consequence, the vertices of the elementary K-cells, $K = 2, \ldots, N$, of the circular lattice and the corresponding vertices of its Ribaucour transform are contained in a K dimensional sphere. Finally, for $K = N$, one obtains a special $\mathbb{Z}^N$ family of N-dimensional spheres, called the Ribaucour congruence of spheres.

Algebraically, the Ribaucour transformation needs only a half of the data (necessary to build the congruence) of the fundamental transformation. The data of the vectorial Ribaucour transformation consists of the solution $\boldsymbol{Y}_i^* : \mathbb{Z}^N \to \mathbb{V}^*$, of the linear system (2.6). Then, because of the circularity constraint, $\boldsymbol{Y}_i : \mathbb{Z}^N \to \mathbb{V}$ given

by

$$\boldsymbol{Y}_i = \left(\boldsymbol{\Omega}(\boldsymbol{X}, \boldsymbol{Y}^*) + T_i \boldsymbol{\Omega}(\boldsymbol{X}, \boldsymbol{Y}^*)\right)^T \boldsymbol{X}_i,$$

is a solution of the linear system (2.7), and the constraints

$$\boldsymbol{\Omega}(\boldsymbol{Y}, H) + \boldsymbol{\Omega}(X^\circ, \boldsymbol{Y}^*)^T = 2\boldsymbol{\Omega}(\boldsymbol{X}, \boldsymbol{Y}^*)^T \boldsymbol{x},$$
$$\boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*) + \boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*)^T = 2\boldsymbol{\Omega}(\boldsymbol{X}, \boldsymbol{Y}^*)^T \boldsymbol{\Omega}(\boldsymbol{X}, \boldsymbol{Y}^*),$$

are admissible.

We remark that the above constraints have a simple geometric meaning when one considers the circular lattices in $\mathbb{E}^M$ as the stereographic projections of quadrilateral lattices in the Möbius sphere S^M; i.e., as a special case of quadrilateral lattices subjected to quadratic constraints.

2.4.2 The symmetric lattice

Given a quadrilateral lattice $\boldsymbol{x}$ with rotation coefficients Q_{ij} and potentials ρ_i given by (2.9), then the functions $\tilde{Q}_{ij}$, defined by equation

$$\rho_j T_j \tilde{Q}_{ij} = \rho_i T_i Q_{ji}, \qquad i \neq j,$$

and called, because of their geometric interpretation, the backward rotation coefficients, satisfy the Darboux system (2.8) as well. A quadrilateral lattice is called *symmetric* if its forward rotation coefficients Q_{ij} are also its backward rotation coefficients. Again the constraint is compatible with the geometric integrability scheme, i.e., it propagates in the construction of the lattice. One can show that a quadrilateral lattice is symmetric if and only if its rotation coefficients satisfy the following trilinear constraint

$$(T_i Q_{ji})(T_j Q_{kj})(T_k Q_{ik}) = (T_j Q_{ij})(T_i Q_{ki})(T_k Q_{jk}), \qquad i, j, k \quad \text{distinct.}$$

To obtain the corresponding reduction of the fundamental transformation we again need only half of the data. Given a solution $\boldsymbol{Y}_i^* : \mathbb{Z}^N \to \mathbb{V}^*$, of the linear system (2.6), then, because of the symmetric constraint, $\boldsymbol{Y}_i : \mathbb{Z}^N \to \mathbb{V}$, defined by

$$\boldsymbol{Y}_i = \rho_i (T_i \boldsymbol{Y}^*)^T,$$

is the solution of the linear system (2.7); notice that, equivalently, we could start from $\boldsymbol{Y}_i$. The constraint

$$\boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*) = \boldsymbol{\Omega}(\boldsymbol{Y}, \boldsymbol{Y}^*)^T,$$

is then admissible and gives a new symmetric lattice.

There are other multidimensional reductions of the quadrilateral lattice like, for example, the D-invariant and Egorov lattices or discrete versions of immersions of spaces of constant negative curvature. We remark that the transformations and reductions discussed above have also a clear interpretation on the level of the analytic methods.

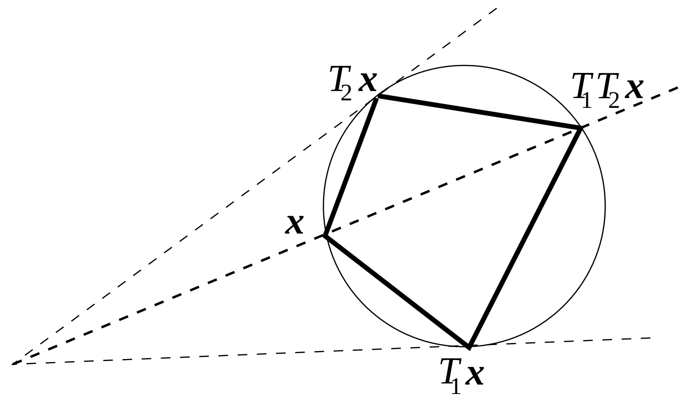

Figure 5: Elementary quadrilaterals of the isothermic lattice

3 Integrable discrete surfaces

In this Section we present some distinguished examples of discrete integrable surfaces. Notice that, although the geometric integrability scheme is meaningless for $N = 2$, however it can be applied indirectly, by considering the discrete surfaces, together with their transformations, as sub-lattices of multidimensional lattices.

We remark also that one can consider integrable evolutions of discrete curves, which give equations of the Ablowitz–Ladik hierarchy, and the corresponding integrable spin chains.

3.1 Discrete isothermic nets

An *isothermic lattice* is a two dimensional circular lattice $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{E}^M$ with harmonic quadrilaterals,; i.e., given $\boldsymbol{x}$, $T_1\boldsymbol{x}$ and $T_2\boldsymbol{x}$, then the point $T_1T_2\boldsymbol{x}$ is the intersection of the circle (passing through $\boldsymbol{x}$, $T_1\boldsymbol{x}$ and $T_2\boldsymbol{x}$) and the line passing through $\boldsymbol{x}$ and the meeting point of the tangents to the circle at $T_1\boldsymbol{x}$ and $T_2\boldsymbol{x}$ (see Figure 5). Therefore, given two discrete curves intersecting in the common vertex $\boldsymbol{x}_0$, the unique isothermic lattice can be found using the above "ruler and compass" construction.

Algebraically the reduction looks as follows. Any oriented plane in $\mathbb{E}^M$ can be identified with the complex plane $\mathbb{C}$. Given any four complex ponts z_1, z_2, z_3 and z_4, their complex cross-ratio is defined by

$$q(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

One can show that the cross-ratio is real if and only if the four points are cocircular or collinear. In particular, a harmonic quadrilateral with vertices numbered anti-clockwise has cross-ratio equal to -1. Therefore, abusing the notation (it can be

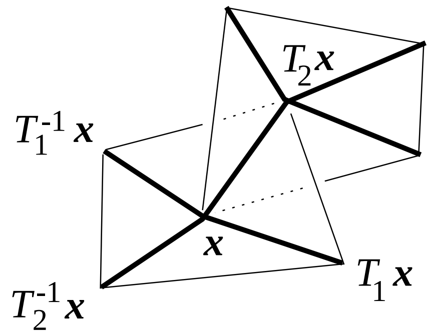

Figure 6: Asymptotic lattices

formalized using Clifford algebras), the isothermic lattice is defined by the condition

$$q(\boldsymbol{x}, T_1\boldsymbol{x}, T_1T_2\boldsymbol{x}, T_2\boldsymbol{x}) = -1.$$

We remark that the definition of isothermic lattices can be slightly generalized allowing for the above cross-ratio to be a ratio of two real functions of single discrete variables.

The restriction of the Ribaucour transformation to the class of isothermic lattices, named after Darboux who constructed it for isothermic surfaces, has as its data a real parameter λ and the starting point $\mathcal{D}(\boldsymbol{x}_0)$, and can be described as follows. Given the elementary quadrilateral $\{\boldsymbol{x}, T_1\boldsymbol{x}, T_2\boldsymbol{x}, T_1T_2\boldsymbol{x}\}$ of the isothermic lattice, and given the point $\mathcal{D}(\boldsymbol{x})$, then the points $\mathcal{D}(T_1\boldsymbol{x})$ and $\mathcal{D}(T_2\boldsymbol{x})$ belong to the corresponding planes and are constructed from equations

$$\begin{aligned} q(\boldsymbol{x}, \mathcal{D}(\boldsymbol{x}), \mathcal{D}(T_1\boldsymbol{x}), T_1\boldsymbol{x}) &= \lambda, \\ q(\boldsymbol{x}, \mathcal{D}(\boldsymbol{x}), \mathcal{D}(T_2\boldsymbol{x}), T_2\boldsymbol{x}) &= -\lambda. \end{aligned}$$

It turns out that the point $\mathcal{D}(T_1T_2\boldsymbol{x})$, constructed by the application of the geometric integrability scheme, is such that the quadrilateral $\{\mathcal{D}(\boldsymbol{x}), \mathcal{D}(T_1\boldsymbol{x}), \mathcal{D}(T_2\boldsymbol{x}), \mathcal{D}(T_1T_2\boldsymbol{x})\}$ is harmonic. Moreover, the construction of the Darboux transformation is compatible; i.e., the new side quadrilaterals have the correct cross-ratios λ and $-\lambda$.

There are various integrable reductions of the isothermic lattice, for example the constant mean curvature lattice and the minimal lattice.

3.2 Asymptotic lattices and their reductions

An *asymptotic lattice* is a mapping $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ such that any point $\boldsymbol{x}$ of the lattice is coplanar with its four nearest neighbours $T_1\boldsymbol{x}$, $T_2\boldsymbol{x}$, $T_1^{-1}\boldsymbol{x}$, $T_2^{-1}\boldsymbol{x}$ (see Figure 6). Such a plane is called the tangent plane of the asymptotic lattice in the point $\boldsymbol{x}$.

It can be shown that any asymptotic lattice $\boldsymbol{x}$ can be recovered from its suitably rescaled normal (to the tangent plane) field $\boldsymbol{N} : \mathbb{Z}^2 \to \mathbb{R}^3$ via the discrete analogue

of the Lelieuvre formulas

$$\Delta_1 \boldsymbol{x} = (T_1\boldsymbol{N}) \times \boldsymbol{N}, \qquad \Delta_2 \boldsymbol{x} = \boldsymbol{N} \times (T_2\boldsymbol{N}). \tag{3.1}$$

By the compatibilty of the Lelieuvre formulas, the normal field $\boldsymbol{N}$ satisfies the discrete Moutard equation

$$T_1T_2\boldsymbol{N} + \boldsymbol{N} = F(T_1\boldsymbol{N} + T_2\boldsymbol{N}), \tag{3.2}$$

for some potential $F : \mathbb{Z}^2 \to \mathbb{R}$.

Given a scalar solution θ of the Moutard equation (3.2), a new solution $\mathcal{M}(\boldsymbol{N})$ of the Moutard equation, with the new potential

$$\mathcal{M}(F) = \frac{(T_1\theta)(T_2\theta)}{(T_1T_2\theta)\theta}F,$$

can be found via the Moutard transformation equations

$$\mathcal{M}(T_1\boldsymbol{N}) \mp \boldsymbol{N} = \frac{\theta}{T_1\theta}(\mathcal{M}(\boldsymbol{N}) \mp T_1\boldsymbol{N}), \tag{3.3}$$

$$\mathcal{M}(T_2\boldsymbol{N}) \pm \boldsymbol{N} = \frac{\theta}{T_2\theta}(\mathcal{M}(\boldsymbol{N}) \pm T_2\boldsymbol{N}). \tag{3.4}$$

Now, via the Lelieuvre formulas (3.1), one can construct a new asymptotic lattice $\mathcal{M}(\boldsymbol{x}) = \boldsymbol{x} \pm \mathcal{M}(\boldsymbol{N}) \times \boldsymbol{N}$. The lines connecting corresponding points of the asymptotic lattices $\boldsymbol{x}$ and $\mathcal{M}(\boldsymbol{x})$ are tangent to both lattices. Such a $\mathbb{Z}^2$-family of lines in $\mathbb{R}^3$ is called Weingarten (or W for short) congruence. Notice that this is not a congruence as considered in Section 2.3.1.

Various integrable reductions of asymptotic lattices are known in the literature: pseudospherical lattices, asymptotic Bianchi lattices and isothermally-asymptotic (or Fubini–Ragazzi) lattices, discrete (proper and improper) affine spheres.

Formally, the Moutard transformation is a reduction of the (projective version of the) fundamental transformation for the Moutard reduction of the Laplace equation. However the geometric relation between asymptotic lattices and quadrilateral lattices is more subtle and the geometric scenery of this connection is the line geometry of Plücker. Straight lines in $\mathbb{R}^3 \subset \mathbb{P}^3$ are considered there as points of the so called Plücker quadric $\mathcal{Q}_P \subset \mathbb{P}^5$. A discrete asymptotic net in $\mathbb{P}^3$, viewed as the envelope of its tangent planes, corresponds to a congruence of isotropic lines in $\mathcal{Q}_P$, whose focal lattices represent the asymptotic directions. The discrete W-congruences are represented by two dimensional quadrilateral lattices in the Plücker quadric.

3.3 The Koenigs lattice

A two dimensional quadrilateral lattice $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{P}^M$ is called a *Koenigs lattice* if, for every point $\boldsymbol{x}$ of the lattice, the six points $\boldsymbol{x}_{\pm 1}$, $T_i\boldsymbol{x}_{\pm 1}$, $T_i^2\boldsymbol{x}_{\pm 1}$, $i = 1, 2$ of its

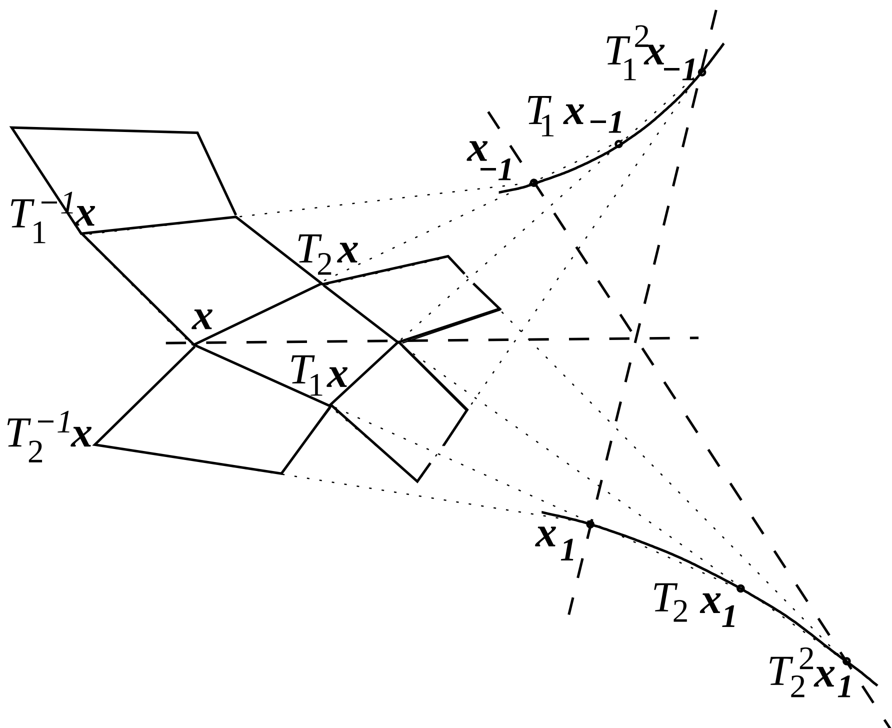

Figure 7: The Koenigs lattice

Laplace transforms (see Section 2.3.3) belong to a conic (see Figure 7). The nonlinear constraint in definition of the Koenigs lattice can be linearized, with the help of the Pascal "mystic hexagon" theorem, to the form that the line passing through $\boldsymbol{x}$ and $T_1T_2\boldsymbol{x}$, the line passing through $\boldsymbol{x}_1$ and $T_1^2\boldsymbol{x}_{-1}$, and the line passing through $\boldsymbol{x}_{-1}$ and $T_2^2\boldsymbol{x}_1$ intersect in a point.

Algebraically, the geometric Koenigs lattice condition means that the Laplace equation of the lattice in homogeneous coordinates $\mathbf{x} : \mathbb{Z}^2 \to \mathbb{R}_*^{M+1}$ can be gauged into the form

$$T_1T_2\mathbf{x} + \mathbf{x} = T_1(F\mathbf{x}) + T_2(F\mathbf{x}). \tag{3.5}$$

It turns out that, if $\boldsymbol{N}$ is a solution of the Moutard equation (3.2), then $\mathbf{x} = T_1\boldsymbol{N} + T_2\boldsymbol{N}$ satisfies the Koenigs lattice equation. Therefore, the algebraic theory of the discrete Koenigs lattice equation (3.5), its (Koenigs) transformation and the permutability of the superpositions of such transformations is based on the corresponding theory for the Moutard equation (3.2).

Geometrically, the Koenigs lattices are selected from the quadrilateral lattices as follows. Given a two dimensional quadrilateral lattice $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{P}^M$ and given a congruence $\mathfrak{l}$ with lines passing through the corresponding points of the lattice. Denote by $\boldsymbol{y}_i = T_i^{-1}\mathfrak{l} \cap \mathfrak{l}$, $i = 1, 2$, points of the focal lattices of the congruence. For every line $\mathfrak{l}$, denote by $\imath$ the unique projective involution exchanging $\boldsymbol{y}_i$ with $T_i\boldsymbol{y}_i$. If, for every congruence $\mathfrak{l}$, the lattice $\mathcal{K}(\boldsymbol{x}) : \mathbb{Z}^2 \to \mathbb{P}^M$, with points $\mathcal{K}(\boldsymbol{x}) = \imath(\boldsymbol{x})$, is a quadrilateral lattice, then the lattice $\boldsymbol{x}$ is a Koenigs lattice. The above construction gives also the corresponding reduction of the fundamental transformation.

A distinguished reduction of the Koenigs lattice is the quadrilateral Bianchi lattice. The natural continuous limit of the corresponding equation is equivalent to the Bianchi (or hyperbolic Ernst) system describing the interaction of planar gravitational waves.

3.4 Discrete two dimensional Schrödinger equation

In the previous secions we have discussed examples of integrable discrete geometries described by equations of hyperbolic type. Below we present some results associated with the elliptic case; it is remarkable that the QL provides a way to connect these two subjects.

Consider a solution $\boldsymbol{N} : \mathbb{Z}^2 \to \mathbb{R}^3$ of the general self-adjoint 5-point scheme on the star of the $\mathbb{Z}^2$ lattice

$$aT_1\boldsymbol{N} + T_1^{-1}(a\boldsymbol{N}) + bT_2\boldsymbol{N} + T_2^{-1}(b\boldsymbol{N}) - c\boldsymbol{N} = 0, \tag{3.6}$$

then the lattice $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ obtained by the Lelieuvre type formulas

$$\Delta_1\boldsymbol{x} = -(T_2^{-1}b)\boldsymbol{N} \times T_2^{-1}\boldsymbol{N}, \qquad \Delta_2\boldsymbol{x} = (T_1^{-1}a)\boldsymbol{N} \times T_1^{-1}\boldsymbol{N}, \tag{3.7}$$

is a quadrilateral lattice having $\boldsymbol{N}$ as normal (to the planes of elementary quadrilaterals) vector field.

The following gauge-equivalent form of equation (3.6)

$$\frac{\Gamma}{T_1\Gamma}T_1\psi + T_1^{-1}(\frac{\Gamma}{T_1\Gamma}\psi) + \frac{\Gamma}{T_2\Gamma}T_2\psi + T_2^{-1}(\frac{\Gamma}{T_2\Gamma}\psi) - q\psi = 0, \tag{3.8}$$

an integrable discretization of the Schrödinger equation

$$\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} - Q\psi = 0,$$

is also the Lax operator associated with an integrable generalization of the Toda law to the square lattice.

The 5-point scheme (3.6) is also a distinguished illustrative example of the sub-lattice theory. Indeed it can be obtained restricting to the even sub-lattice $\mathbb{Z}^2_e$ the discrete Cauchy–Riemann equations

$$T_1T_2\phi - \phi = iG(T_1\phi - T_2\phi). \tag{3.9}$$

Because of the equivalence (on the discrete level!) between equation (3.9) and the discrete Moutard equation (3.2), the 5-point scheme (3.6) inherits integrability properties (Darboux-type transformations, superposition formulas, analytic methods of solution) from the corresponding (and simpler) integrability properties of the discrete Moutard equation.

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List of Figures

See also

Bäcklund transformations
$\bar{\partial}$ approach to integrable systems
Discrete integrable systems
Integrable systems and algebraic geometry
KP equations and geometry
Nonlinear Schrödinger equations
Sine-Gordon equation
Toda lattices

Keywords

Integrable discrete geometry
Quadrilateral lattice
Line congruences
Fundamental transformation
Circular lattice
Symmetric lattice
Isothermic lattices
Asymptotic lattices
Koenigs lattice
Discrete 2D Schrödinger equation